Eccentric Two-Target Model for Qualitative Air Combat Game Analysis

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An air combat duel with "all-aspect" guided missiles between a Harrier type and a faster conventional fighter aircraft is modeled as a two-target game between a "homicidal chauffeur" and an "aggressive pedestrian." The firing envelopes of the missiles are approximated by eccentric circles in the faster airplane's coordinate system. The qualitative two-target game analysis is a nontrivial combination of two zero-sum pursuit-evasion games with eccentric circular target sets. The analysis confirms the need for large "off-boresight" angle missiles for the less maneuverable aircraft and the great sensitivity of the solution to firing envelope parameters.

Nomenclature

A	= aspect angle
a,b	= eccentricity of the fast and slow aircraft capture circle, respectively
$\mathcal{F}$	= fast aircraft
$\mathcal{H}$	= slow aircraft (Harrier type)
L	= off-boresight angle
R	= best turning radius of the fast aircraft
S	= switch function of the fast aircraft control
t	= time
V_1, V_2	= velocity of the fast and slow aircraft, respectively
x,y	= Cartesian coordinates in the coordinate frame of the fast aircraft
α,β	= radius of the fast and slow aircraft capture circle, respectively
γ	= velocity ratio $(V_2/V_1)<1$
δ	= parameter in the terminal manifold
θ	= angular parameter of the target sets
λ_x,λ_y	= costate variables
τ	= retrograde time, $=t_f-t$
ϕ,ψ	= control of the fast and slow aircraft, respectively

Subscripts

a,b	= fast and slow aircraft target set, respectively
f	= final value
p	= value at the intersection of the target sets
s	= value at the "stagnation point"
$u\alpha,u\beta$	= boundary of the usable part of the fast and slow aircraft target set, respectively

Introduction

DIFFERENTIAL game models of air combat have been based on the understanding that this is the appropriate mathematical framework to analyze the conflicting features of such a scenario. Most works in this area followed the approach of Isaacs[1] using a perfect information zero-sum pursuit-evasion formulation. Several simplified dynamic models were investigated,[2-6] leading to some useful qualitative conclusions. Unfortunately, the zero-sum pursuit-evasion formulation does not suit well the most important air-to-air combat problem: the engagement between two aggressively motivated fighter aircraft. An a priori selection of the roles (pursuer vs evader), which is implicitly included in the game formulation, does not take place in such a scenario. Consequently, the notion of "role determination,"[7] assuming that one of the pilots should accept the role of evader from the outset, seems to be of limited value. The conceptual complexity of the problem has been acknowledged and discussed.[8] A meaningful investigation of such engagements requires a different formulation based on the two-target game concept.[9]

In a two-target differential game, each player has a different target set in the common game space. The objective of each player is to drive the game from a given initial condition to its own target set, while avoiding the target set of the opponent. Such formulation has an inherently qualitative (game of kind) nature, defining in the common state space the "winning zones" of the players. The game can also terminate by a "mutual kill" if both target sets are reached simultaneously, or by an inconclusive "draw" if neither of the targets is reached in a finite time.

The solution of any differential game is always sensitive to the target set modeling. For two-target games this sensitivity is even more crucial and has an overwhelming influence on the solution. In the role determination analysis,[7] a line segment aligned with the velocity vector was used to model the classical air-to-air weapon of the aircraft—the gun. In several recent papers using two-target game formulations,[10-12] guided missile firing envelopes were modeled by concentric circular target sets. Unfortunately, such terminal surfaces, selected for their analytical simplicity, do not represent properly the salient characteristics of real modern weapon systems.

The major weapon system of a modern fighter aircraft in air-to-air combat consists of "all-aspect" guided missiles. These missiles can be fired in any direction relative to the target and from a relatively large domain of "off-boresight" angles. The maximum effective firing ranges of these missiles, which determine the respective target sets of a differential game, are a function of the relative geometry depicted in Fig. 1.

The monotonic dependence of the effective firing range on the target aspect angle A and the off-boresight angle L implies that firing envelopes of modern air-to-air missiles can be approximately described by eccentric circles. Moreover, it can be shown that the respective eccentricities play a decisive role in the game analysis.

Full representation of aircraft dynamics leads to a very high-dimensional game model, which is prohibitive for an analytical investigation. Simplified game models frequently have been used for such purposes. One of the simplest models

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is that of the "homicidal chauffeur."[2,10,11] In this planar, constant-speed model, one of the players is an aggressive driver of a constant-speed (V_1) automobile having a finite turning radius R. The second player is a "pedestrian" with a lower speed ($V_2 < V_1$) but with an instantaneous turning capability. Although such a game model seems to be certainly oversimplified, it contains the main features of an air-to-air combat between a Harrier ($\mathcal{H}$) and a faster conventional fighter ($\mathcal{F}$), as has taken place recently in the Falkland Islands war.

The objective of this paper is a qualitative game analysis of such an air combat scenario with "all-aspect" guided missiles. The analysis is based on a two-target formulation of the "homicidal chauffeur" game,[10-12] using eccentric circular terminal surfaces. As an introductory step toward the two-target game solution, two independent zero-sum pursuit-evasion games are solved.

The Eccentric "Homicidal Chauffeur" Game

This game is similar to the original version.[1,2] The game dynamics is expressed in relative coordinates with respect to the faster airplane $\mathcal{F}$ using nondimensional variables. Distances are normalized by "R," the best turning radius of $\mathcal{F}$, assumed to have a unit speed $V_1 = 1.0$. The speed of the slower aircraft $\mathcal{H}$ is $V_2 = \gamma < 1.0$. This aircraft is assumed to have an instantaneous turning rate. Consequently, its motion is controlled by its relative heading ψ. The control variable of $\mathcal{F}$ is its normalized turning rate ϕ bounded by $-1 \le \phi \le 1$. The nondimensional equations of the relative motion are

$$\dot{x} = -y\phi + \gamma\sin\psi \tag{1}$$

$$\dot{y} = x\phi - 1 + \gamma\cos\psi \tag{2}$$

The target set of the game is an eccentric circle of radius α as shown in Fig. 2, the center of the circle being ahead of $\mathcal{F}$ and along the y-axis by a normalized distance $a < \alpha$. This target set is a simplified representation of the firing envelope of the missile carried by $\mathcal{F}$. The eccentricity indicates the loss of effective range in the case of an off-boresight launch due to the turning required of the missile. The equation of the target circle,

$$x_f^2 + (y_f - a)^2 = \alpha^2 \tag{3}$$

can be expressed in parametric form

$$x_f = \alpha\sin\theta \tag{4}$$

$$y_f = \alpha\cos\theta + a \tag{5}$$

Due to symmetry only the right half-plane of the game space $0 \le \theta \le \pi$ is investigated.

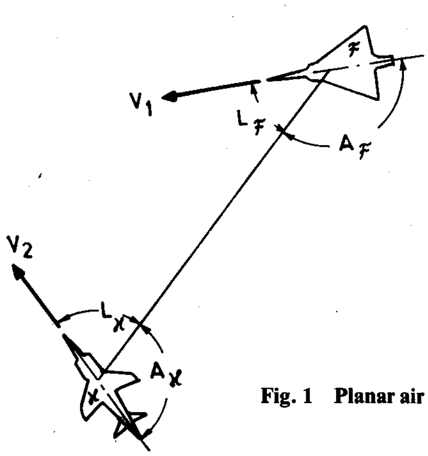

Fig. 1 Planar air combat geometry.

The boundary of the usable part (BUP) of the target set is defined by the value of $\theta_{u\alpha} > 0$. This value is determined, following Isaac's approach,[1] by

$$\min_{\phi_f}\max_{\psi_f}[-a\sin\theta_{u\alpha}\phi_f + \gamma(\sin\psi_f\sin\theta_{u\alpha} + \cos\psi_f\cos\theta_{u\alpha}) - \cos\theta_{u\alpha}] = 0 \tag{6}$$

leading to the optimal values (denoted by an overbar) of the terminal controls strategies

$$\bar{\phi}_f = 1 \tag{7}$$

$$\bar{\psi}_f = \theta_{u\alpha} \tag{8}$$

and to

$$\theta_{u\alpha} = \cos^{-1}\{[\gamma - a(1 + a^2 - \gamma^2)^{1/2}]/(1 + a^2)\} \tag{9}$$

The usable part of the target set, therefore, is given by $|\theta| < \theta_{u\alpha}$. This last statement implicitly assumes that the weapon system modeled by the target set has indeed the appropriate "off-boresight" capability. Note also that for $a = 0$, Eq. (9) reduces the classical result[1] $\cos\theta_{u\alpha} = \gamma$.

The Hamiltonian of the game is

$$H = \lambda_x(-y\phi + \gamma\sin\psi) + \lambda_y(x\phi - 1 + \cos\psi) \tag{10}$$

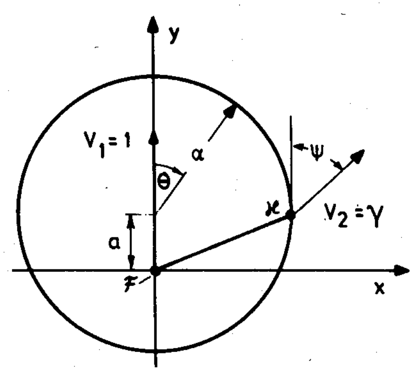

Fig. 2 Eccentric target set for the pursuer.

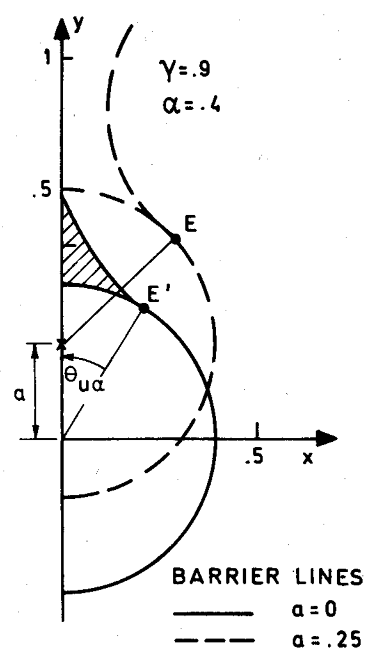

Fig. 3 Eccentric "homicidal chauffeur" game.

where λ_x and λ_y are the costate variables satisfying the adjoint differential equations

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -\lambda_y \bar{\phi} \tag{11}$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y} = \lambda_x \bar{\phi} \tag{12}$$

The optimal control strategies $\bar{\phi}$ and $\bar{\psi}$ are obtained from the main equation of Isaacs[1] $\min_{\phi}\max_{\psi} H = 0$, which leads to

$$\bar{\phi} = -\text{sign}(-y\lambda_x + x\lambda_y) \triangleq -\text{sign}S \tag{13}$$

$$\bar{\psi} = \tan^{-1}(\lambda_x/\lambda_y) \tag{14}$$

Retrograde integration of the optimal trajectory emanating from the BUP determines the "barrier"[1] of the game. If the barrier is closed, the game space is divided into a "capture zone" and an "escape zone" for the evader H. If the barrier is open, capture is guaranteed from any initial condition.

The transversality conditions on the target set require

$$\lambda_x(t_f) = \sin\delta \tag{15}$$

$$\lambda_y(t_f) = \cos\delta \tag{16}$$

where δ is a parameter to be determined by satisfying the main equation. It was shown[2] that for $a=0$ the barrier is tangent to the target circle if $\alpha^2 \leq 1-\gamma^2$. For an eccentric target ($a>0$), similar consideration leads to the condition

$$\alpha^2 \leq 1 + a^2 - \gamma^2 \tag{17}$$

If the inequality (17) is satisfied, then $\delta = \theta_{u\alpha}$, and the final value of the switch function is obtained by combining Eqs. (4), (5), (13), (15), and (16)

$$S_f = -a\sin\theta_{u\alpha} \tag{18}$$

leading to $\bar{\phi}_f = 1$.

Simultaneous retrograde integration of the state and costate equations with $\tau = (t_f - t)$ from the BUP leads to

$$\bar{\psi}(\tau) = \tan^{-1}(\lambda_x/\lambda_y) = (\tau + \theta_{u\alpha}) \tag{19}$$

and consequently

$$x(\tau) = a\sin\tau - \cos\tau + 1 + (\alpha - \gamma\tau)\sin(\tau + \theta_{u\alpha}) \tag{20}$$

$$y(\tau) = a\cos\tau + \sin\tau + (\alpha - \gamma\tau)\cos(\tau + \theta_{u\alpha}) \tag{21}$$

The conditions to be satisfied for barrier closure on the y-axis are expressed by the following inequality:

$$\alpha \leq (1 + a^2 - \gamma^2)^{1/2} + \gamma\sin^{-1}(\cos\theta_{u\alpha}) - 1 \tag{22}$$

If Eq. (22) is not satisfied, the open barrier terminates when the switch function S becomes zero for the first time. This time is given by

$$\tau^* = 2\pi - \cos^{-1}(\gamma) - \theta_{u\alpha} \tag{23}$$

In Figs. 3 and 4, the effect of the eccentricity parameter "a" on the barrier closure is illustrated. In the example depicted in Fig. 3 it is shown that the barrier, which is closed for a concentric terminal surface ($a=0$), is open for $a=0.25$. The α vs γ relationship leading to a tangentially closed barrier is depicted in Fig. 4 for constant values of the eccentricity parameter a. It can be seen that even a subcritical eccentricity leads to an enlarged usable part of the target set and, consequently, to an enlarged capture zone.

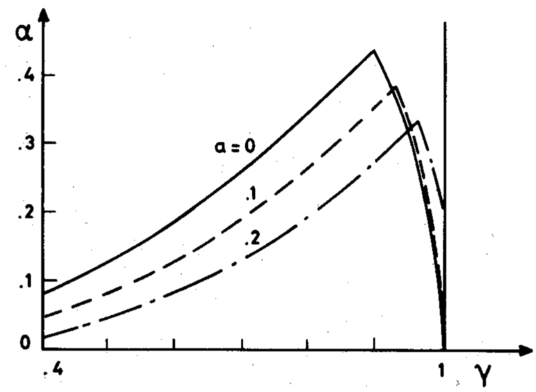

Fig. 4 Conditions for barrier closure.

The above analysis is performed for the cases where inequality [Eq. (17)] is satisfied. If the target radius α is large enough to violate Eq. (17), the semipermeable surface emanating from the BUP cannot be tangent to the capture circle. Moreover, it was demonstrated[2] in this case that the semipermeable surface cannot be an optimal trajectory.

If Eq. (17) is violated, a barrier can exist only when the game has a "stagnation point" on the capture circle, defined by $\dot{x}_f = \dot{y}_f = 0$. In this situation, both aircraft fly on concentric circles (in the real space) with equal turning rates keeping the distance of separation constant. Such a "stagnation point" can exist if the following inequality is satisfied:

$$(\alpha - \gamma)^2 < 1 + a^2 < (\alpha + \gamma)^2 \tag{24}$$

The geometrical interpretation of Eq. (24) is that the capture circle can be intersected by another circle of radius γ centered at (1,0) as shown in Fig. 5.

The coordinates of the "stagnation point" are given by the equations

$$x_s = (M + N + aQ)/2M \tag{25}$$

$$y_s = [a(M-n) + Q]/2M \tag{26}$$

where

$$M = 1 + a^2, \quad N = \alpha^2 - \gamma^2, \quad Q - [4\alpha^2 M - (N+M)^2]^{1/2} \tag{27}$$

The optimal control strategies at the "stagnation point" that can be considered as a point of "safe contact,"[2] are the following:

$$\bar{\phi}_s = 1 \tag{28}$$

$$\bar{\psi}_s = \tan^{-1}[y_s/(1 - x_s)] \tag{29}$$

Using these terminal strategies the equations of motion can be integrated in retrograde time τ yielding for the barrier coordinates

$$x(\tau) = y_s\sin\tau + (x_s - 1)\cos\tau + 1 - \gamma\tau\sin(\tau + \bar{\psi}_s) \tag{30}$$

$$y(\tau) = y_s\cos\tau - (x_s - 1)\sin\tau - \gamma\tau\cos(\tau + \bar{\psi}_s) \tag{31}$$

This barrier is always open and ends when the switch function is equal to zero for the first time, which is given by

$$\tau^* = 2\pi - \cos^{-1}(\gamma) - \bar{\psi}_s \tag{32}$$

This barrier locally separates regions of different control strategies.

It is to be noted that the existence of a "stagnation point" does not exclude the existence of a tangent barrier. If the value of the eccentricity satisfies $\alpha^2+\gamma^2<1+a^2<(\alpha+\gamma)^2$ both types of barriers exist. It can be shown that the two barriers do not intersect. This completes the qualitative analysis of the eccentric "homicidal chauffeur" game.

The Game of the Aggressive Pedestrian

In this game the pursuer is the slower aircraft $\mathcal{H}$ wishing to capture its faster opponent $\mathcal{F}$. The equations of motion are the same as in the previous game. In this game the target set is defined in the coordinate frame of the evader and represents the "vulnerability zone" of the fast aircraft $\mathcal{F}$ with respect to the missile launched by its instantaneously turning opponent. Such target set is an eccentric circle, as explained in detail in the Appendix and depicted in Fig. 6. Its equation in parametric form is

$$x_f=\beta\sin\theta \tag{33}$$

$$y_f=\beta\cos\theta+b \tag{34}$$

As in the previous game, symmetry allows us to concentrate on the right half-plane only $(0\leq\theta\leq\pi)$.

The BUP of the target set in this game is determined from

$$\min_{\psi_f}\max_{\phi_f}\,[-b\sin\theta_{u\beta}\phi_f+\gamma(\sin\psi_f\sin\theta_{u\beta}+\cos\psi_f\cos\theta_{u\beta})-\cos\theta_{u\beta}]=0 \tag{35}$$

yielding

$$\bar{\phi}_f=-1 \tag{36}$$

$$\bar{\psi}_f=\pi+\theta_{u\beta} \tag{37}$$

and

$$\theta_{u\beta}=\cos^{-1}\{[-\gamma+b(1+b^2-\gamma^2)^{1/2}]/(1+b^2)\} \tag{38}$$

The usable part of the target set is evidently $|\theta|<\theta_{u\beta}$.

The semipermeable surface emanating from the BUP is a "barrier." It has to satisfy the main equation where H is identical to the expression given in Eq. (10). The optimal control strategies along the barrier of this game are

$$\bar{\phi}=\text{sign}[-y\lambda_x+x\lambda_y]\triangleq\text{sign}S \tag{39}$$

$$\bar{\psi}=\tan^{-1}(-\lambda_x/-\lambda_y)=\pi+\tan^{-1}[\lambda_x/\lambda_y] \tag{40}$$

i.e., they are of opposite sign compared to the strategies of the previous game given by Eqs. (13) and (14).

Satisfaction of the main equation at t_f requires that

$$\lambda_x(t_f)=\sin\theta_{u\beta} \tag{41}$$

$$\lambda_y(t_f)=\cos\theta_{u\beta} \tag{42}$$

leading to

$$S_f=-b\sin\theta_{u\beta}<0 \tag{43}$$

and, consequently, $\bar{\phi}_f=-1$.

Retrograde integration of the equations of motion yields the coordinates of the barrier

$$x(\tau)=-b\sin\tau+\cos\tau-1+(\gamma\tau+\beta)\sin(\theta_{u\beta}-\tau) \tag{44}$$

$$y(\tau)=b\cos\tau+\sin\tau+(\gamma\tau+\beta)\cos(\theta_{u\beta}-\tau) \tag{45}$$

It is easy to verify that, starting with Eq. (43), the switch function remains negative for all $x(\tau)>0$ and there is no value of τ to satisfy $x(\tau)=\dot{x}(\tau)=0$ simultaneously. Therefore, it can be concluded that for the present game the barrier is always closed. The faster evader $\mathcal{F}$ can be captured only from a limited set of initial conditions.

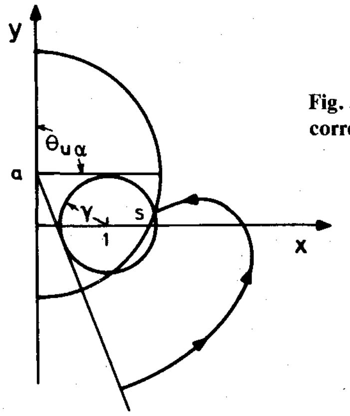

Fig. 5 "Stagnation point" and the corresponding barrier.

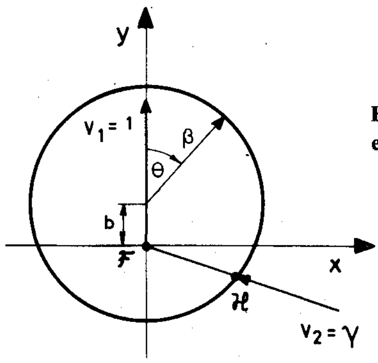

Fig. 6 Eccentric target set in evader coordinates.

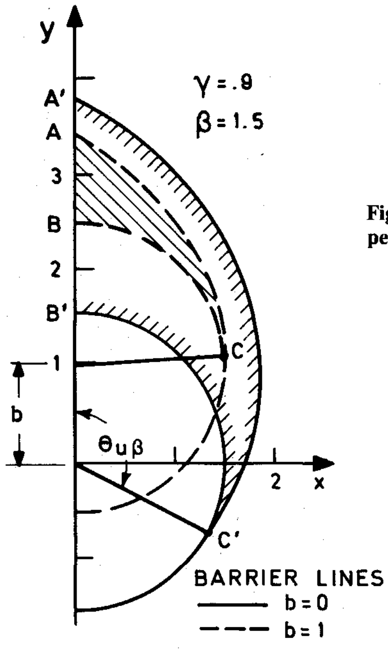

Fig. 7 Eccentric "aggressive pedestrian" game.

In Figs. 7 and 8 the influence of the eccentricity parameter $(0\leq b\leq\beta)$ on the capture zone is shown. As it may be seen in Fig. 7, the capture zone for a concentric target set $(b=0)$ is the largest. If the game starts in this region, $\mathcal{F}$ cannot turn away rapidly enough to benefit from its higher speed for escaping. The eccentricity of the target set, due to the fact that the firing range from a tail aspect is much shorter than in a "head-on" encounter, improves the situation of the turn-limited evader. The larger the value of the eccentricity parameter b, the smaller the usable part of the target set $\theta_{u\beta}$. Consequently, the corresponding capture zone shrinks (see Fig. 7).

The solution of the "aggressive pedestrian" game clearly indicates the potential of a slow but highly maneuverable aircraft equipped with an "all-aspect" missile.

Two-Target Game Analysis

The first phase in a two-target game analysis is the overlay of the respective target sets. This step may lead to one of the following results:

1) One target set is completely enclosed by the other one.

2) The usable part of one target set is completely enclosed by the opponent's target.

3) The target sets are completely disjoint.

4) The target sets intersect but both UP's are outside the opponent's target.

5) The target sets intersect and one UP is outside the opponent's target.

6) The UP's of the respective target sets intersect.

In cases 1 and 2, the two target game is completely dominated by one of the players and only one pursuit-evasion game has to be solved. In cases 3 and 4, the next step would be to solve two separate pursuit-evasion games of kind. If both games have closed barriers these must be disjoint, and the two-target game is solved automatically. The game space is decomposed into three parts: the two "winning zones" enclosed by the respective barriers, and the complement of their union, the "draw" region.

If, however, one of the barriers in case 3 or 4 is open, or if cases 5 and 6 apply, the two-target game becomes complex and certainly nontrivial.

For the air combat game analyzed in the present paper cases 3 and 4 do not exist; therefore, we concentrate on cases 5 and 6, where at least one player's UP is reduced. In any particular case the intersection of the target sets determines the "effective UP's" of the respective players.

The solution of the game of kind, i.e., determination of the "winning zones," depends for the presently used simplified model on five parameters. An extensive investigation of the problem in the entire parameter space ($<\gamma\leq 1$, $0<\alpha<\infty$, $0<\beta<\infty$, $0\leq a<\alpha$, $0\leq b<\beta$) is out of the scope of the present paper. Attention is therefore focused on analyzing three examples representing air-to-air duels between a Harrier ($\mathcal{H}$) and a faster conventional fighter aircraft ($\mathcal{F}$), both equipped with "all-aspect" guided missiles. The parameters are selected to provide a realistic model for an actual scenario. The speed ratio $\gamma=0.75$ seems to indicate the inherent limitation of the subsonic Harrier compared to a supersonic opponent. Since the firing range of modern air-to-air missiles greatly exceeds the turning radius of a supersonic fighter, in the first two examples the values of $\alpha=2.3$ and $\beta=2.25$ were selected. The eccentricity parameters of the two target sets have to be quite different. For the faster airplane the eccentricity represents the dependence of the firing range on the "off-boresight" angle, while with respect to the Harrier-type aircraft it expresses the effect of the closing velocity. Since for all known missile models the second effect is more important, in any practical example, $b>a$. In the first two examples, $a=1.3$ and $b=1.6$ were chosen as representative values.

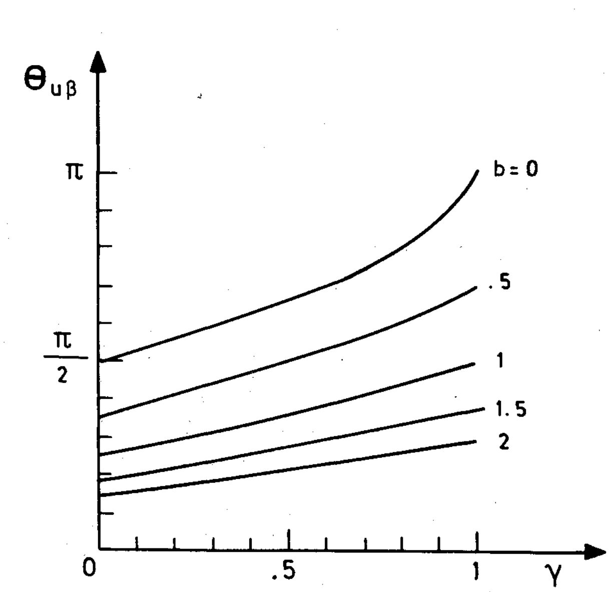

Fig. 8 Effect of eccentricity on the usable part of the target set ("aggressive pedestrian" game).

The two examples differ only in the value of the maximum admissible off-boresight angle for the missile of $\mathcal{F}$. In the first case, there are no inherent limitations, and the missile can be fired even again at a target flying behind the launching aircraft (bomber defense). In the second case, the maximum value of the "off-boresight launch" angle is limited to ± 60 deg.

As has been indicated already, the analysis starts with the overlay of the respective target sets. In the first two examples, they intersect at the point P as shown in Fig. 19. P is also a point of "mutual kill." The intersection point can be also characterized by the target set parameters θ_a and θ_b, respectively. For the present examples the target set of $\mathcal{H}$ covers a great part of $\mathcal{F}$'s target. Since the intersection point P falls inside the usable part of $\mathcal{F}$'s target, $\theta_a<\theta_{u\alpha}$, the arc PE is still usable. At the other end, the usable part of $\mathcal{H}$'s target set satisfies $\theta_b>\theta_{u\beta}$. Consequently, the "winning zone" of $\mathcal{H}$ is its capture zone ABC in the "aggressive pedestrian" game and the segment CP (not including the point P).

For the faster aircraft $\mathcal{F}$ the situation has dramatically changed in comparison to the respective pursuit-evasion game. Although $\mathcal{F}$ can force its slower opponent from anywhere toward the original target set, the trajectories which terminate not on the arc PE lead to the victory of $\mathcal{H}$. To identify the "winning zone" of $\mathcal{F}$ in this two-target game a semipermeable surface has to be constructred from P.

The equation of this semipermeable surface is certainly different from the one in the "homicidal chauffeur" game given by Eqs. (20-21). The main difference is in the value of the parameter δ involved in the transversality conditions. In the present case at the intersection point P

$$\delta_p=\cos^{-1}\{[y_p(x_p-1)^2+y_p^2-\gamma^2)^{1/2} -\gamma(x_p-1)]/[(x_p-1)^2+y_p^2]\} \tag{46}$$

The switch function of $\mathcal{F}$ at the point P is given by

$$S_f=-\gamma+\cos\delta_p \tag{47}$$

and it is negative as long as

$$y_p\leq\gamma\{\gamma a+[\alpha^2-a^2(1-\gamma^2)]^{1/2}\} \tag{48}$$

Equation (46) can be expressed in an alternative form using the notation of θ_a.

$$\theta_a\geq\cos^{-1}\{\gamma[1-(1-\gamma^2)a^2/\alpha^2]^{1/2} -[(1-\gamma^2)a/\alpha]\}\triangleq\theta_a^* \tag{49}$$

If this last inequality is satisfied, then $\bar{\phi}_f=1$, and the semipermeable surface emanating from P is a barrier. In our examples $\theta_a=76.7$ deg and $\theta_a^*=63.4$ deg, therefore, Eq. (49) is indeed satisfied. The coordinates of the barrier trajectory

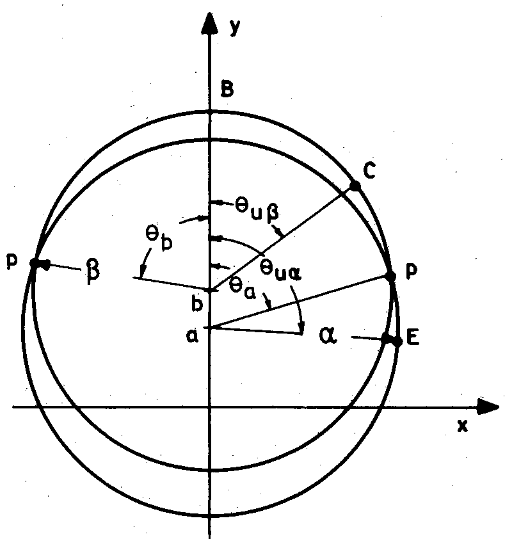

Fig. 9 Overlay of two eccentric target sets.

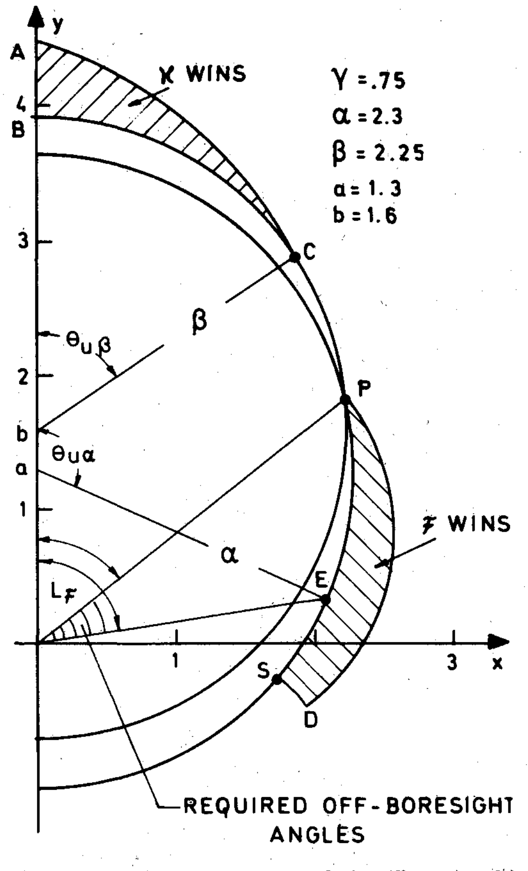

Fig. 10 Eccentric two-target game solution (first example).

are obtained by retrograde integration, yielding

$$x(\tau) = y_p \sin\tau + (x_p - l)\cos\tau + l - \gamma\tau\sin(\tau + \delta_p) \tag{50}$$

$$y(\tau) = y_p \cos\tau - (x_p - l)\sin\tau - \gamma\tau\cos(\tau + \delta_p) \tag{51}$$

Such a barrier can either end on the target set of $\mathcal{F}$ or intersect another semipermeable surface emanating from an eventual "stagnation point." In the first example a stagnation point does exist, as shown in Fig. 10. Consequently, the winning zone of $\mathcal{F}$ is limited to the closed region PSD. Trajectories starting inside this zone (or entering into it due to an error of the opponent) will terminate in the usable part PE charac-

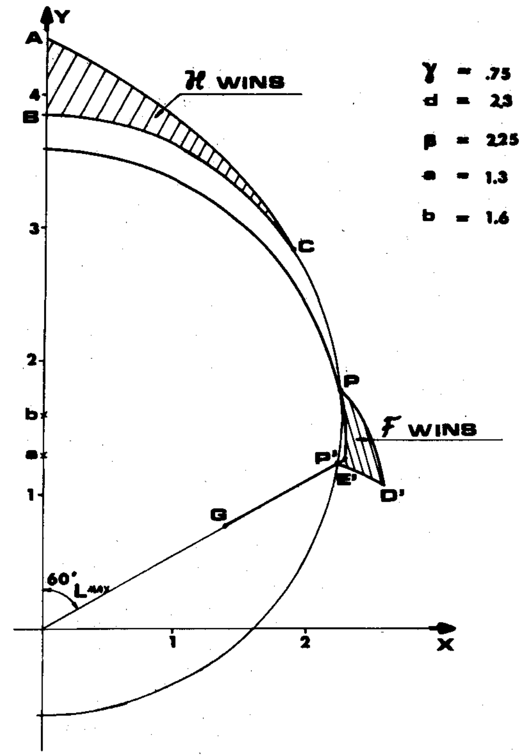

Fig. 11 Eccentric two-target game solution (second example).

terized by $\theta_a \leq \theta_{u\alpha}$. In practical terms, this means that the weapon system of $\mathcal{F}$ should have a rather extended "off-boresight" launch capability in order to validate the above-presented solution.

The effect of an eventual limitation of the admissible "off-boresight" launch angle is demonstrated in the next example assuming $|L_{\max}| = 60$ deg. In this case, depicted in Fig. 11, the UP of $\mathcal{F}$'s target set contains the circular arc until E′ and the line segment E′G on the "off-boresight" limit. First, there is no stagnation point on this target set. The line segment E′G intersects the target circle of $\mathcal{H}$ at P′, and, therefore, only the section P′E′ is effectively usable. The effective UP of $\mathcal{F}$'s target set is therefore PE′P′. The barriers constructed from the two intersection points P and P′ meet at the point D′ (see Fig. 11), defining a much smaller "winning zone" for $\mathcal{F}$ than in the previous case shown in Fig. 10.

In the present qualitative analysis optimal strategies are uniquely determined only along the barriers. Inside the respective "winning zones" they can be arbitrarily chosen or determined uniquely as a solution of some quantitative pursuit-evasion game (time-optimal, for example). Outside the "winning zones" the outcome of the two-target game is inconclusive. Neither of the players can force its optimally evading opponent inside the capture zone. It is very likely that in this "draw" region both aircraft will fly aggressively. When such a trajectory (which is certainly *not optimal* in the usual pursuit-evasion game sense) approaches either one of the target sets, or the boundaries of one of the "winning zones," the pilot of the aircraft which is threatened should apply its optimal evasive strategy. If the "homicidal chauffeur" game does not have a closed barrier, as in our examples, the dynamics in the "draw" region of the two-target game is dominated by the faster airplane. $\mathcal{F}$ can drive all trajectories towards its originally usable target set $0 < \theta \leq \theta_{u\alpha}$ against any maneuver of $\mathcal{H}$, but cannot enforce penetration into the "winning zone" if $\mathcal{H}$ evades optimally. If a trajectory comes near to the segments PC or CA (or P′G), it is $\mathcal{F}$ that has to switch to the optimal evasive strategy to avoid the victory of $\mathcal{H}$. Trajectories ending by a "mutual kill" at P and P′ can be avoided by both aircraft.

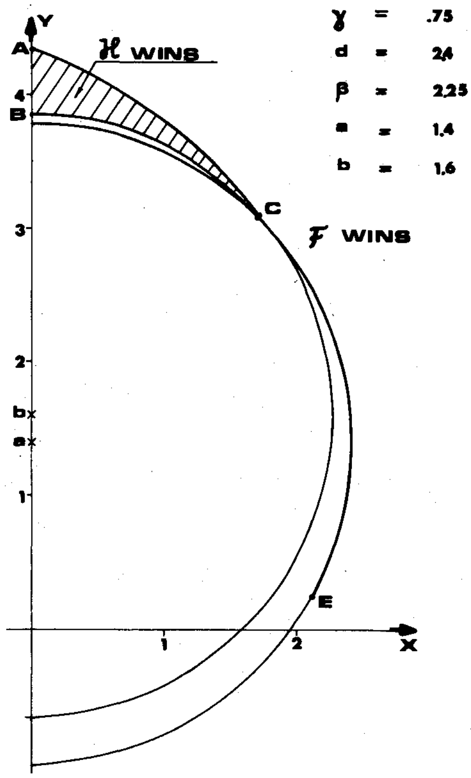

Fig. 12 Eccentric two-target game solution (third example).

The purpose of the third example is to demonstrate the great sensitivity of the two-target game solution to the firing envelope parameters. In this example the values of $\alpha=2.4$, $a=1.4$ were selected, indicating a slight improvement in $\mathfrak{F}$'s missile, compared to the first example ($\alpha=2.3$, $a=1.3$). The result of such an improvement is quite astonishing. In this example, depicted in Fig. 12, the target circles intersect at C inside the UP of $\mathfrak{H}$, and consequently, its "winning zone" is reduced. Moreover, the effective UP of $\mathfrak{F}$ is now CE. Since the only barrier emanating from the boundaries of this UP is CA, there is no "draw" region for this game. The entire game space, excluding the "winning zone" ABC of $\mathfrak{H}$, is the "winning zone" of $\mathfrak{F}$ (if no "off-boresight" limitation is assumed). Note that C is also a point of "mutual kill."

Concluding Remarks

In this paper an attempt is made to analyze a relevant air combat duel by using a simple dynamic model and a realistic approximation of the weapon envelopes in a two-target game formulation. The results of the qualitative analysis, which led to determining the "winning zones" of the respective aircraft, depend on five independent nondimensional parameters. A systematic investigation of the game in the parameter space being out of the scope of the present paper, the main features of the two-target game analysis are demonstrated by using three representative examples.

In spite of the limitations (simple dynamic model, few examples) the reported qualitative game analysis implies some practical conclusions.

1) In "head-on" encounters, followed by a firing exchange of similar weapons, there is an advantage for the more maneuverable aircraft even if it is slower.

2) The faster and less maneuverable aircraft can dominate the air combat in the "draw" region only if it switches its role and strategy very carefully. A uniformly aggressive behavior is certaintly not beneficial.

3) In order to compete successfully with a highly maneuverable opponent equipped with an "all-aspect" missile, a large "off-boresight" launch angle of the weapon system is needed.

4) The game solution is very sensitive to the firing envelope parameters.

These conclusions, confirmed by many full-scale simulation studies, illustrate the validity of a simplified two-target game model for air combat analysis and encourage further investigation in this direction.

Appendix: Target Set of the "Aggressive Pedestrian"

The coordinate system of this game is centered at the position of the fast evader and aligned with its velocity vector. The pursuer is equipped with an "all-aspect" missile assumed to be fired at the ideal collision course. The instantaneous turning capability of the slower pursuer is consistent with this assumption. For such a missile the effective time of flight t_M and the corresponding distance flown with respect to the ground R_M are determined by design considerations and the launching velocity V_2.

Based on these assumptions the maximum firing range of such a missile against a straight-flying target strongly depends on the target aspect angle A_F defined to be between the line of sight and the target direction (see Fig. 1). For a "head-on" firing the maximum range is

$$R(A_F=180\,\mathrm{deg})=R_M+V_1t_M \tag{A1}$$

while for a tail launch

$$R(A_F=0\,\mathrm{deg})=R_M-V_1t_M \tag{A2}$$

From simple geometric considerations it is easy to see that for any value of A_F

$$R(A_F)=[R_M^2-(V_1t_M\sin A_F)^2]^{1/2}-(V_1t_M\cos A_F) \tag{A3}$$

This is the equation of a circle of the radius R_M centered at the point ($y=V_1t_M$) ahead of the target as shown in Fig. 6. Therefore, $\beta=R_M$ and $b=V_1t_M$.

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